Instability in the Quantum Helicon Dispersion Relation*

N. DAVID MERMIN AND V. CELLI
University of California, San Diego, La Jolla, California
(Received 28 May 1964)

The classical stability theorem for the conventional thermal equilibrium state of an electron gas in a uniform magnetic field is generalized to (a) the semiclassical case, (b) the quantum case when there is no magnetic field, and (c) the quantum case in the presence of a magnetic field, provided that only Coulomb interactions are retained. However, when the quantum gas in a magnetic field is treated with all electromagnetic interactions, at very low temperatures it becomes unstable against transverse excitations propagating in the direction of the field. This instability appears as a root of the quantum helicon dispersion relation in the upper half frequency plane. It is shown that the instability is due to the failure of the conventional Hartree ground state (in which the one-electron states are the ordinary Landau ones) to minimize the ground-state energy, when magnetic current-current interactions are retained along with Coulomb interactions. We have found a state giving a lower energy than the conventional one, in which transverse volume currents exist perpendicular to the magnetic field. Because, however, the magnetic coupling is very weak, the reduction in energy is unobservably small at any realistic field strengths or electronic densities. We conclude that the instability does not lead to any measurable effects, and that for all practical purposes the conventional thermal equilibrium state can be regarded as stable.

I. INTRODUCTION

We would like to call attention to and explain the significance of an instability in the quantum mechanical helicon dispersion relation. We shall show in a self-consistent field approximation[1] that, at sufficiently low temperatures, the conventional equilibrium state (a Fermi distributions of electrons in Landau levels) of an electron gas in a uniform magnetic field is unstable against the formation of transverse current waves traveling along the direction of the field; corresponding to this dynamic instability, we shall show that there is a Hartree ground state of an electron gas in a uniform magnetic field with lower energy than the conventional state, in which volume currents are present perpendicular to the field and oscillating in amplitude along the direction of the field. These facts would appear to cast doubt on theories of the equilibrium behavior of electrons in magnetic fields. However, we shall also argue that the energy difference between the unorthodox and conventional ground states is so minute as to render the instability of no practical significance. Our purpose is therefore not to report any new physical effects, but to explain the instability in the helicon dispersion relation, and to show that in spite of it the physical transverse excitations propagating along the magnetic field are stable.

The instability occurs in the linear response of the conventional equilibrium state of the electrons in a static magnetic field to an electromagnetic disturbance which is calculated self-consistently, i.e., the sources of which are just the mean currents and charge densities it induces. The occurrence of the instability is rather surprising, since the corresponding classical problem is known to be stable. From rather general free-energy considerations, Newcomb[2] showed that the combined system of Maxwell's equations and the Boltzmann-Vlasov equation in a magnetic field, linearized about a Boltzmann distribution, has no solutions that grow in time. In Appendix A we show that Newcomb's argument can be generalized to the following cases: (a) classical electrons in a magnetic field with a Fermi equilibrium distribution, (b) quantum electrons in a magnetic field interacting only through self-consistently calculated Coulomb forces, and (c) quantum electrons interacting through the full set of Maxwell's equations but in the absence of a static magnetic field. However, in the case of quantum electrons in a magnetic field with all electromagnetic interactions retained, the Newcomb argument produces not a proof of stability but a condition on temperature, density, and field strength sufficient to ensure stability. For transverse modes propagating along the field, this condition is necessary and sufficient, and the instability in the helicon dispersion relation is revealed by its failure.

From the variety of circumstances under which one *can* prove stability theorems, one can infer quite a bit about the nature of the instability. It must be a quantum effect, it requires the presence of a magnetic field, and is due to electromagnetic interactions other than Coulomb. This suggests that one reexamine the Hartree variational principle for electrons in a magnetic field interacting not only through Coulomb, but also through magnetic current-current interactions. A particular Hartree state is the familiar electrically neutral, volume current-free state, consisting of a Fermi distribution of electrons in Landau levels. We shall show that the instability is due to the existence of other Hartree states containing transverse volume currents perpendicular to

* Supported in part by the U. S. Office of Naval Research.

[1] All results of this paper are found within a self-consistent field approximation, although we will generally not repeat this restriction with each assertion. We shall also ignore electron spin, since to take it into account would only complicate the analysis without altering any conclusions.

[2] The theorem is proved in an appendix to I. B. Bernstein, Phys. Rev. 109, 10 (1958).

the magnetic field, with lower energy than the conventional one.

These states are quite similar to the nonuniform Hartree states found by Overhauser[3] in the one-dimensional Fermi system with attractive interactions. Kohn and Nettel[4] argued that if the attractive interactions are weak such a ground state exists only in one dimension. The instability in the helicon dispersion relation is an Overhauser effect of this kind. The static magnetic field, by quantizing motion in the perpendicular plane, provides the one dimensionality necessary if the effect is to occur for weak interactions, and the very weak magnetic interaction between parallel currents provides the attraction.

In Sec. II we derive a necessary and sufficient condition for the stability of transverse excitations propagating along the field, which we use in Sec. III to show that instabilities always exist at low enough temperatures. A rough calculation of the transition temperature reveals it to be extremely low, foreshadowing the conclusion of Sec. IV that the instability is due to the existence of Hartree states with lower—but negligibly lower—energy than the conventional ground state.

Throughout we shall be working with a gas of N electrons in a box of volume V, in the presence of a uniform background of positive charge of density $n_0=N/V$. The magnetic field $\mathbf{B}_0$ will be taken to be in the z direction, and described by the vector potential

$$\mathbf{A}_0=(0,B_0x,0). \tag{1.1}$$

We shall take the quantization box to have dimensions L_0 in the z direction, and L in the x and y directions. We record here the definitions

$$\omega_p=(4\pi n_0e^2/m)^{1/2}, \tag{1.2}$$

$$\omega_c=|e|B_0/mc, \tag{1.3}$$

$$F(E)=1/(e^{\beta(E-\mu)}+1),\quad \beta=1/k_BT, \tag{1.4}$$

and

$$f_n(p)=F(p^2/2m+(n+\tfrac{1}{2})\omega_c). \tag{1.5}$$

II. NECESSARY AND SUFFICIENT STABILITY CONDITION

An electron gas can sustain transverse normal modes propagating parallel to the applied magnetic field at frequencies and wavelengths satisfying[5,6]

$$\omega^2=\omega_p^2+k^2c^2+\left(\frac{\omega_p^2}{n_0}\right)m\omega_c\sum_{n=0}^{\infty}(n+1)\omega_c \times\int\frac{dp}{(2\pi)^2}\frac{f_n(p-\frac{1}{2}k)-f_{n+1}(p+\frac{1}{2}k)}{\omega-\omega_c-pk/m}. \tag{2.1}$$

When $\omega_c\ll\omega_p$, at sufficiently long wavelengths (2.1) has a root at the helicon (or whistler) frequency, $\omega=\omega_ck^2c^2/\omega_p^2$. For want of a more convenient designation we shall refer to (2.1) as the helicon dispersion relation, but we must stress that it only describes the helicon mode in certain limits, and that it describes other modes as well, such as the transverse plasmons at $\omega=\pm(\omega_p^2+k^2c^2)^{1/2}$.

A growing wave made up of circularly polarized transverse currents can occur if (2.1) has any roots in the upper half ω plane. In this section we shall prove that at any nonzero temperature (2.1) has no roots in the upper half plane provided

$$\omega_p^2+k^2c^2\geqslant\left(\frac{\omega_p^2}{n_0}\right)m\omega_c\sum_n(n+1)\omega_c \times\int\frac{dp}{(2\pi)^2}\frac{f_n(p-\frac{1}{2}k)-f_{n+1}(p+\frac{1}{2}k)}{\omega_c+pk/m}, \tag{2.2}$$

and otherwise exactly one such root.

To avoid repetition of lengthy formulas in our proof, we define

$$P(\omega)=(m\omega_c^2\omega_p^2/4\pi^2n_0)\sum_n(n+1) \times[f_n(p-\tfrac{1}{2}k)-f_{n+1}(p+\tfrac{1}{2}k)]\delta(\omega-\omega_c-pk/m), \tag{2.3}$$

in terms of which the dispersion relation (2.1) is just

$$g(\omega)\equiv\omega_p^2+k^2c^2-\omega^2+\int d\bar{\omega}P(\bar{\omega})/(\omega-\bar{\omega})=0. \tag{2.1$'$}$$

The stability condition (2.2) is that[7] $g(0)\geqslant0$, so we must prove that g has no zeros in the upper half-plane if and only if it is non-negative at the origin. The only properties of P needed for the proof are

(a) $\omega P(\omega)>0,\quad \omega\neq0$;
(b) $\int d\omega P(\omega)=\omega_p^2\omega_c$;
(c) $P(\omega)$ vanishes rapidly as $\omega\to\infty$.

Property (a) follows from (2.3) and the fact that the Fermi distribution is a decreasing function of energy. The strong inequality (which holds only at nonzero temperatures) is essential. Property (b) follows from the fact that

$$n_0=m\omega_c\sum_n\int dp f_n(p)/(2\pi)^2. \tag{2.4}$$

Property (c) refers to the fact that P vanishes as a Gaussian for large ω, and will be appealed to in order to justify taking certain limits inside integrals.

We now look for zeros of g in the upper half-plane. Let $\omega=re^{i\theta}$, $0<r<\infty$, $0<\theta<\pi$. The real and imaginary parts of g are

$$\mathrm{Re}g(re^{i\theta})=\omega_p^2+k^2c^2-r^2(2\cos^2\theta-1) +\int d\bar{\omega}(P(\bar{\omega})/D)(r\cos\theta-\bar{\omega}), \tag{2.5}$$

[3] A. W. Overhauser, Phys. Rev. Letters 4, 415 (1960).
[4] W. Kohn and J. Nettel, Phys. Rev. Letters 5, 8 (1960).
[5] J. J. Quinn and S. Rodriguez, Phys. Rev. 128, 2487 (1962).
[6] V. Celli and N. D. Mermin (to be published).
[7] Since $P(0)=0$, $g(0)$ is defined and independent of how zero is approached.

$$\text{Im}g(re^{i\theta})=-r\sin\theta\left[2r\cos\theta+\int d\bar{\omega}P(\bar{\omega})/D\right],\qquad(2.6)$$

where the denominator D is

$$D(\bar{\omega},r,\theta)=|\omega-\bar{\omega}|^2=r^2+\bar{\omega}^2-2r\bar{\omega}\cos\theta.\qquad(2.7)$$

$\text{Im}(g)$ will vanish in the upper half-plane whenever

$$2r\cos\theta=-\int d\bar{\omega}P(\bar{\omega})/D.\qquad(2.8)$$

Consider Eq. (2.8) for a given positive r. As $\cos\theta$ goes from -1 to 1, the left side increases from $-2r$ to $2r$. The right side approaches $+\infty$ as $\cos\theta\to-1$, and $-\infty$ as $\cos\theta\to1$, due to property (a). Furthermore, its derivative with respect to $\cos\theta$

$$-2r\int d\bar{\omega}\bar{\omega}P(\bar{\omega})/D^2,$$

is negative. There is therefore exactly one solution, $\theta_0(r)$, between 0 and π, for each value of $r>0$. Clearly, θ_0 will be a continuous differentiable function of r. Furthermore, as $r\to\infty$, from (b) and (c),

$$2r\cos\theta_0\sim\omega_p{}^2\omega_c/r^2.\qquad(2.9)$$

Thus, $g(\omega)$ is real along a curve in the upper half-plane which starts at the origin and asymptotically approaches the imaginary axis as $r\to\infty$, and is real nowhere else in the upper half-plane. Along this curve (2.8) holds, so

$$\text{Re}g(re^{i\theta_0(r)})=\omega_p{}^2+k^2c^2-r^2(4\cos^2\theta_0(r)-1)-\int d\bar{\omega}\bar{\omega}P(\bar{\omega})/D.\qquad(2.10)$$

From (2.10) and (2.9),

$$\lim_{r\to\infty}\text{Re}g(re^{i\theta_0(r)})=\infty.$$

Since $\text{Re}(g)$ is continuous along the curve it will therefore vanish at least once on the curve if it is negative as $r\to0$. Thus ,when $g(0)<0$, $g(\omega)$ has at least one zero in the upper half-plane. To establish the rest of the theorem—that it has only one zero, and that when $g(0)\geq0$ there are no zeros in the upper half-plane—it suffices to show that $g(re^{i\theta_0(r)})$ is an increasing function of r.

From (2.10) and (2.7),

$$\tfrac{1}{2}(\partial/\partial r)\,\text{Re}g(re^{i\theta_0(r)})=r-4r\cos\theta_0\partial(r\cos\theta_0)/\partial r+\int d\bar{\omega}(\bar{\omega}P/D^2)[r-\bar{\omega}\partial(r\cos\theta_0)/\partial r].\qquad(2.11)$$

Differentiating (2.8), which implicitly gives θ_0 as a function of r, we find

$$r\int d\bar{\omega}(P/D^2)=\left[1+\int d\bar{\omega}(\bar{\omega}P/D^2)\right]\times\partial(r\cos\theta_0(r))/\partial r.\qquad(2.12)$$

Finally, from (2.7) and (2.8), we have

$$\int d\bar{\omega}(\bar{\omega}^2P/D^2)+r^2\int d\bar{\omega}(P/D^2)=-2r\cos\theta_0\left[1-\int d\bar{\omega}(\bar{\omega}P/D^2)\right].\qquad(2.13)$$

If we use (2.12) and (2.13) to eliminate in (2.11) the occurrence of $\int d\bar{\omega}(\bar{\omega}^2P/D^2)$ in favor of $\int d\bar{\omega}(\bar{\omega}P/D^2)$, we find that

$$\tfrac{1}{2}(\partial/\partial r)\,\text{Re}g(re^{i\theta_0(r)})=r\sin^2\theta_0[1+(r\partial\theta_0/\partial r)^2]\times\left[1+\int d\bar{\omega}(\bar{\omega}P/D^2)\right],\qquad(2.14)$$

which is positive by virtue of property (a).

This completes the proof that (2.2) is a necessary and sufficient condition for stability of the helicon dispersion relation at nonzero temperatures.

III. FAILURE OF STABILITY CONDITION

Although the stability condition (2.2) has been derived only for nonzero temperatures, if it fails at zero temperature, we can be sure of finding instabilities at very low temperatures as long as the left side of (2.2) is continuous at $T=0$. When the temperature is zero, (2.2) becomes

$$\omega_p{}^2+k^2c^2\geq\frac{m^2\omega_c{}^2\omega_p{}^2}{(2\pi)^2n_0}\sum_{n=0}^{n_{\max}}(2/k)\ln\left|\frac{p_n+p_{n+1}+k}{p_n+p_{n+1}-k}\right|,\qquad(3.1)$$

where p_n is the Fermi momentum for the nth Landau level,

$$p_n=[2m(\mu-(n+\tfrac{1}{2})\omega_c)]^{1/2},\qquad(3.2)$$

and $n_{\max}$ is the quantum number of the highest occupied level,

$$(n_{\max}+\tfrac{1}{2})\omega_c<\mu<(n_{\max}+\tfrac{3}{2})\omega_c.\qquad(3.3)$$

Evidently (3.1) fails for k sufficiently close to p_n+p_{n+1} for any $n<n_{\max}$, whatever the magnetic field strength or density.

It is easy to verify that things are continuous at $T=0$. We consider for algebraic simplicity the case $k=p_n+p_{n+1}$, $n<n_{\max}$, which leads to the worst violation of (3.1), and show that for such k at any nonzero magnetic field strength and density there are instabilities at sufficiently low temperatures. Since each term in the n summation in (2.2) is positive, when $k=p_n+p_{n+1}$ we

keep only the nth term to get a condition necessary for stability[8]:

$$\omega_p{}^2+(p_n+p_{n+1})^2c^2 \geqslant \frac{m^2\omega_c{}^2\omega_p{}^2(n+1)}{(2\pi)^2 n_0(p_n+p_{n+1})} \times\left[P\int\frac{dpf_n(p)}{p_n-p}+P\int\frac{dpf_{n+1}(p)}{p_{n+1}-p}\right]. \quad (3.4)$$

Integrating by parts we find, in the limit as $\beta p_n{}^2/2m\rightarrow\infty$,

$$P\int\frac{dpf_n(p)}{p_n-p}=\ln(\beta p_n{}^2/m)-\int\frac{dx\ln|x|}{2\cosh^2 x}. \quad (3.5)$$

The remaining integral is of order 1. Clearly then, for sufficiently low temperatures there will be instabilities when $k=p_n+p_{n+1}$. It will be easiest to reverse the inequality in (3.4) when p_n+p_{n+1} is as small as possible, i.e., when the $(n+1)$th Landau level is barely occupied, so that $p_n{}^2/2m=\omega_c$, $p_{n+1}=0$. Under these conditions $n\omega_c\approx\mu$, the energy of the last occupied state, which is roughly independent of B_0 at fixed density. The inequality in (3.4) will be reversed when

$$\pi\sqrt{2}(\omega_p{}^2+2mc^2\omega_c)/e^2\mu(m\omega_c)^{1/2}\approx\ln(\beta\omega_c). \quad (3.6)$$

Because of the energy mc^2, the left side of (3.6) is quite large. It is least for given density when $\omega_c=\omega_p{}^2/2mc^2$; for this value of B_0 (roughly a few kilogauss at metallic densities and decreasing linearly with decreasing density), (3.6) becomes

$$\ln(\beta\omega_c)\approx 4\pi(137)\omega_p/\mu. \quad (3.7)$$

Since ω_p/μ is somewhat larger than unity at metallic densities and varies only as $n_0{}^{-1/6}$, the transition temperature given by (3.7) is far too low to be of any physical significance.

With the aid of (2.4) one can estimate the size of the terms omitted from the right side of (3.4) and argue that these cannot alter the right side of (3.7) by the several orders of magnitude necessary to lead to an observable transition temperature. Rather than go into this further, we shall now show the cause of the instability and explain directly why it occurs only at unobservably low temperatures.

IV. HARTREE GROUND STATE

The helicon dispersion relation is derived by calculating the linear response of an electron gas initially in thermal equilibrium to a time-dependent electromagnetic disturbance. The thermal equilibrium state is taken to be a Fermi distribution of electrons in eigenstates of the one-electron Hamiltonian

$$\mathcal{H}_0=\tfrac{1}{2}mv_0{}^2,$$

where

$$\mathbf{v}_0=(\mathbf{p}-e\mathbf{A}_0/c)/m.$$

The eigenstates are the Landau functions,[9]

$$\psi_{np_zp_y}=(LL_0)^{-1/2}e^{i(p_yy+p_zz)}\phi_n(x+p_y/m\omega_c), \quad (4.1)$$

where the ϕ_n are the orthonormal wave functions for a one-dimensional harmonic oscillator of mass m and frequency ω_c. The corresponding eigenvalues are

$$E_{np_z}=p_z{}^2/2m+(n+\tfrac{1}{2})\omega_c, \quad (4.2)$$

with degeneracy $m\omega_cL^2/2\pi$.

The basic significance of the helicon instability is that in a self-consistent field approximation a simple Fermi distribution of electrons in Landau states does not give the lowest free energy.[10] To avoid irrelevant complications, we shall consider only the situation at zero temperature, and shall show that the conventional Hartree ground state does not minimize the expectation value of the energy in a self-consistent field approximation. In fact, there are a variety of states, all of which give a lower Hartree energy than the conventional

$$E_0=\frac{m\omega_cL^2}{2\pi}\sum_{n=0}^{n_{\max}}\sum_{p_z=-p_n}^{p_n}\left(\frac{p_z{}^2}{2m}+(n+\tfrac{1}{2})\omega_c\right).$$

Since, however, all suffer from the limitation that the energy reduction is unobservably small, it seems a waste of effort to discuss them systematically. We shall therefore examine only a typical class of such states. The kinds of refinements one can use to produce states of still lower energy should then be clear; it is our (unproved) belief that no more clever choice will increase the energy reduction by the factor of about 10^{300} necessary to make it observable.

The conventional Hartree ground state Ψ_0 is a product of one-electron Landau states, containing $\psi_{np_yp_z}$ if $E_{np_z}\leqslant\mu$, or, equivalently, if $-p_n\leqslant p_z\leqslant p_n$. Now consider any two adjacent Landau cylinders. (By a cylinder we mean the set of all states with a given n.) We construct a trial state Ψ by letting all cylinders except these two contribute to Ψ in the conventional way. For these two cylinders, however, we occupy the states

$$\begin{aligned}\phi_{nqp_y}{}^{(1)}&=a_q\psi_{n,p_n+q,p_y}+b_q\psi_{n+1,-p_{n+1}+q,p_y}, \quad -p_n\leqslant q<p_{n+1},\\ \phi_{nqp_y}{}^{(2)}&=a_q\psi_{n,-p_n-q,p_y}+b_q\psi_{n+1,p_{n+1}-q,p_y}, \quad -p_n<q\leqslant p_{n+1}, \quad (4.3)\\ &|a_q|^2+|b_q|^2=1.\end{aligned}$$

The states ϕ are an orthonormal set and are orthogonal

---

[8] In (2.2) the numerator of the integrand vanishes whenever the denominator does. The singular integrals in (3.4) were introduced by separately treating the two terms in the numerator of (2.2), and can be interpreted as principal value integrals.

[9] L. D. Landau and E. M. Lifschitz, *Quantum Mechanics* (Pergamon Press, Inc., New York, 1958), p. 474.

[10] The conventional state is a solution to the Hartree equations with uniform density, zero volume currents, and hence vanishing self-consistent fields. It does, of course, have surface currents, but the effect of these is already included if we regard $\mathbf{B}_0$ as the true internal field, generated by external sources and by the surface currents.

to all other occupied Landau levels, so the many-particle wave function Ψ is an acceptable Hartree wave function. The range of values of q in (4.3) is such as to produce the same number of one-electron states as in the conventional wave function. Indeed, if we make the choice

$$\begin{aligned} a_q&=0 \quad (b_q=1), \quad q>0, \\ a_q&=1 \quad (b_q=0), \quad q<0, \end{aligned} \tag{4.4}$$

then Ψ reduces to the conventional state, Ψ_0. We shall show that this choice of the a's does not lead to the lowest energy.

It is simplest to discuss the Hartree approximation in terms of the one-particle density matrix, φ. If the trial Hartree wave function is

$$\Psi(\mathbf{r}_1,\cdots,\mathbf{r}_N)=\prod_\alpha \phi_\alpha(\mathbf{r}_\alpha)\,,$$

then φ is defined to be

$$\langle\mathbf{r}|\varphi|\mathbf{r}'\rangle=\sum_\alpha \phi_\alpha(\mathbf{r})\phi_\alpha{}^*(\mathbf{r}')\,. \tag{4.5}$$

The energy in a Hartree state is the electron kinetic energy plus the energy of the self-consistent fields:

$$E=\mathrm{tr}\tfrac{1}{2}mv^2\varphi+\int d\mathbf{r}(E_1{}^2+B_1{}^2)/8\pi\,, \tag{4.6}$$

where $\mathbf{v}$ is the velocity operator,

$$\mathbf{v}=(\mathbf{p}-e(\mathbf{A}_0+\mathbf{A}_1)/c)/m=\mathbf{v}_0-e\mathbf{A}_1/mc\,, \tag{4.7}$$

and $\mathbf{E}_1$ and $\mathbf{B}_1$ satisfy

$$\nabla\cdot\mathbf{E}_1=4\pi(\rho-\rho_0)=4\pi e(\langle\mathbf{r}|\varphi|\mathbf{r}\rangle-n_0)\,, \tag{4.8}$$

$$\nabla\times\mathbf{B}_1=4\pi\mathbf{j}/c=(4\pi e/c)\langle\mathbf{r}|\tfrac{1}{2}\{\mathbf{v},\varphi\}|\mathbf{r}\rangle\,. \tag{4.9}$$

(The curly bracket is an anticommutator.)

We first solve (4.8) and (4.9) to eliminate the self-consistent fields from (4.6), leaving an energy that explicitly depends only on φ. One easily verifies that in the state Ψ the density remains uniform and equal to $n_0=N/V$, so that $\mathbf{E}_1=0$. To calculate $\mathbf{B}_1$ it is useful to have an expression for the currents present in the state Ψ in the absence of a self-consistent vector potential:

$$\mathbf{j}_0(\mathbf{r})=e\langle\mathbf{r}|\tfrac{1}{2}\{\varphi,\mathbf{v}_0\}|\mathbf{r}\rangle\,. \tag{4.10}$$

The set of single-particle states (4.3) continues to give a vanishing component of $\mathbf{j}_0$ parallel to $\mathbf{B}_0$, since it is invariant under a reflection in the $x-y$ plane. There is now, however, a nonvanishing volume current perpendicular to $\mathbf{B}_0$, given by

$$\begin{aligned} j_0{}^{\pm}(\mathbf{k})&=2^{-1/2}\int d\mathbf{r}e^{-i\mathbf{k}\cdot\mathbf{r}}[j_{0x}(\mathbf{r})\pm ij_{0y}(\mathbf{r})] \\ &=2^{-3/2}e\int d\mathbf{r}e^{-i\mathbf{k}\cdot\mathbf{r}}\langle\mathbf{r}|\{v_x\pm iv_y,\ \varphi\}|\mathbf{r}\rangle \\ &=\tfrac{1}{2}e\ \mathrm{tr}\{v_0{}^{\pm},\varphi\}e^{-i\mathbf{k}\cdot\mathbf{r}} \\ &=\tfrac{1}{2}e\ \mathrm{tr}\varphi\{v_0{}^{\pm},e^{-i\mathbf{k}\cdot\mathbf{r}}\}\,. \end{aligned} \tag{4.11}$$

This can also be written as

$$\begin{aligned} j_0{}^{\pm}(\mathbf{k})&=e\ \mathrm{tr}\varphi e^{-i\mathbf{k}\cdot\mathbf{r}}v_0{}^{\pm}-k^{\pm}\rho(\mathbf{k})/m \\ &=e\ \mathrm{tr}\varphi e^{-i\mathbf{k}\cdot\mathbf{r}}v_0{}^{\pm}\,, \end{aligned} \tag{4.12}$$

since $\rho(\mathbf{k})=0$ unless $\mathbf{k}=0$. It is enough to calculate $\mathbf{j}_0{}^+(\mathbf{k})$, since

$$j_0{}^-(\mathbf{k})=j_0{}^+(-\mathbf{k})^*\,. \tag{4.13}$$

For this purpose, we use the following property of the Landau functions:

$$v_0{}^+\psi_{n,p_z,p_y}=[(n+1)\omega_c/m]^{1/2}e^{i\varphi}\psi_{n+1,p_z,p_y}\,. \tag{4.14}$$

(The particular value of the phase factor will turn out to be of no consequence.) Since only the nth and $(n+1)$th levels contribute to $\mathbf{j}_0$, we have

$$\begin{aligned} j_0{}^+(\mathbf{k})&=e\sum_{p_y,q,\lambda}(\phi_{nqp_y}{}^{(\lambda)},e^{-i\mathbf{k}\cdot\mathbf{r}}v_0{}^+\phi_{nqp_y}{}^{(\lambda)}) \\ &=e[(n+1)\omega_c/m]^{1/2}e^{i\varphi}\sum_{qp_y}b_q{}^*a_q\Big[\int d\mathbf{r}\psi_{n,p_n+q,p_y}{}^*e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_{n,-p_{n+1}+q,p_y}+(p_n,p_{n+1},q)\to(-p_n,-p_{n+1},-q)\Big] \\ &=e[(n+1)\omega_c/m]^{1/2}e^{i\varphi}(m\omega_cV/2\pi L_0)\textstyle\sum_q(b_q{}^*a_q)\delta(k_x,0)\delta(k_y,0)[\delta(k_z,p_n+p_{n+1})+\delta(-k_z,p_n+p_{n+1})]\,. \end{aligned} \tag{4.15}$$

We absorb the phase factor $e^{i\varphi}$ into a_q. (Evidently, its value determines the direction of $\mathbf{j}_0$ in the $x-y$ plane.)

We can write E in terms of $\mathbf{j}_0$, since

$$\begin{aligned} \mathrm{tr}\varphi(\tfrac{1}{2}mv^2)&=\mathrm{tr}\varphi\mathcal{H}_0-(e/4c)\ \mathrm{tr}\varphi\{(\mathbf{v}+\mathbf{v}_0),A_1\} \\ &=\mathrm{tr}\varphi\mathcal{H}_0-(2c)^{-1}\int d\mathbf{r}(\mathbf{j}+\mathbf{j}_0)\cdot\mathbf{A}_1\,. \end{aligned} \tag{4.16}$$

But

$$\int d\mathbf{r}B_1{}^2/8\pi=\int d\mathbf{r}(\mathbf{A}_1\cdot\mathbf{j})/2c\,, \tag{4.17}$$

so

$$E=\mathrm{tr}\varphi\mathcal{H}_0-\int d\mathbf{r}(\mathbf{A}_1\cdot\mathbf{j}_0)/2c\,. \tag{4.18}$$

Furthermore, in the gauge $\nabla\cdot\mathbf{A}_1=0$,

$$-\nabla^2\mathbf{A}_1=4\pi\mathbf{j}_0/c-(4\pi e/c)\langle\mathbf{r}|e\varphi\mathbf{A}_1/mc|\mathbf{r}\rangle$$
$$=4\pi\mathbf{j}_0/c+4\pi n_0e^2\mathbf{A}_1/mc^2, \quad (4.19)$$

since φ gives a uniform density. Thus,

$$E=\mathrm{tr}\,\varphi\mathcal{H}_0-(2\pi/V)\times\sum_{\mathbf{k}}(k^2c^2+\omega_p^2)^{-1}\mathbf{j}_0(\mathbf{k})\cdot\mathbf{j}_0(-\mathbf{k}), \quad (4.20)$$

or, from (4.15)

$$E=\mathrm{tr}\,\varphi\mathcal{H}_0-\frac{(2V/\pi L_0^2)(n+1)m\omega_c^3e^2}{(p_n+p_{n+1})^2c^2+\omega_p^2}|\sum_q b_q^*a_q|^2. \quad (4.21)$$

We wish to compare (4.21) with the conventional ground state energy, $\mathrm{tr}\,\varphi^0\mathcal{H}_0$. The contribution to $\mathrm{tr}(\varphi-\varphi^0)\mathcal{H}_0$ comes only from the nth and $(n+1)$th levels. The conventional state gives

$$\frac{m\omega_cL^2}{2\pi}\left[\sum_{p_z=-p_n}^{n}\left(\frac{p_z^2}{2m}+(n+\tfrac{1}{2})\omega_c\right)+\sum_{p_z=-p_{n+1}}^{p_{n+1}}\left(\frac{p_z^2}{2m}+(n+\tfrac{3}{2})\omega_c\right)\right], \quad (4.22)$$

while the corresponding contribution from φ is

$$\frac{m\omega_cL^2}{\pi}\sum_{q=-p_n}^{p_n+}\left[|a_q|^2\left(\frac{(p_n+q)^2}{2m}+(n+\tfrac{1}{2})\omega_c\right)+|b_q|^2\left(\frac{(p_{n+1}-q)^2}{2m}+(n+\tfrac{3}{2})\omega_c\right)\right]. \quad (4.23)$$

Subtracting (4.22) from (4.23) gives

$$(L^2/\pi)(p_n+p_{n+1})\omega_c\left[\sum_{0}^{p_{n+1}}q|a_q|^2-\sum_{-p_n}^{0}q|b_q|^2\right]. \quad (4.24)$$

Equations (4.24) and (4.21) give as the energy difference per unit volume between the states Ψ and Ψ_0,

$$(E-E_0)/V=(\omega_c(p_n+p_{n+1})/2\pi^2)\times\left[\frac{2\pi}{L_0}\left(\sum_{0}^{p_{n+1}}q|a_q|^2-\sum_{-p_n}^{0}q|b_q|^2\right)-g\left|\frac{2\pi}{L_0}\sum_{-p_n}^{p_{n+1}}a_q^*b_q\right|^2\right], \quad (4.25)$$

where g is the dimensionless quantity

$$g=(n+1)me^2\omega_c^2/\pi(p_n+p_{n+1})\times(\omega_p^2+(p_n+p_{n+1})^2c^2). \quad (4.26)$$

In the limit of infinite volume $((2\pi/L_0)\sum_q\rightarrow\int dq)$ Eq. (4.25) can be written

$$(E-E_0)/V=[\omega_c(p_n+p_{n+1})/2\pi^2]\times\left[\tfrac{1}{2}p_n^2+\int_{-p_n}^{p_{n+1}}q|a_q|^2dq-g\left|\int_{-p_n}^{p_{n+1}}a_q^*b_qdq\right|^2\right]. \quad (4.27)$$

Evidently if we wish to minimize (4.27), we can take a_q and b_q to be real. It is then a straightforward variational calculation to make (4.27) stationary subject to the constraint $a_q^2+b_q^2=1$. One finds that $(E-E_0)/V$ is stationary when

$$a_q^2=\tfrac{1}{2}[1-q/(q^2+q_0^2)^{1/2}], \quad (4.28)$$

where

$$q_0^2=[p_{n+1}^2+p_n^2+2p_np_{n+1}\cosh(1/g)]/\sinh^2(1/g). \quad (4.29)$$

This choice of a_q leads to

$$(E-E_0)/V=-(\omega_c(p_n+p_{n+1})/4\pi^2)\times[\tfrac{1}{2}(p_{n+1}^2+p_n^2)(\coth(1/g)-1)+p_{n+1}p_n/\sinh(1/g)], \quad (4.30)$$

which is in fact negative for any value of g.

The effective coupling constant g is largest when the $(n+1)$th level is just starting to be occupied, i.e., when $n=n_{\max}$ and $p_n^2/2m=\omega_c$. Under these conditions, (4.26) becomes

$$g=(2\pi)^{-1}(137)^{-1}(n+1)\omega_c\times(2mc^2\omega_c)^{1/2}/(2mc^2\omega_c+\omega_p^2). \quad (4.31)$$

As in the preceding section, $(n_{\max}+1)\omega_c\approx\mu$, independent of ω_c, so g is largest when $\omega_c=\omega_p^2/2mc^2$, where it has the value

$$g=(4\pi)^{-1}(137)^{-1}(\mu/\omega_p). \quad (4.32)$$

Placing this in (4.30) we see that $(E-E_0)/V$, though negative, is absurdly small in magnitude. Comparing (4.32) and (4.30) with (3.7), we see that the instability in the helicon dispersion relation appears only at temperatures so low that thermal energies do not obscure the difference between E and E_0.

We conclude that the helicon instability is due to the existence of a Hartree wave function giving lower ground-state energy than the conventional one, but by an immeasurably small amount.[11] There are, it is true, better choices than (4.3) which give still lower energies. One could, for example, extend the coupling (4.3) to

[11] Indeed, the quantity q_0 [Eq. (4.29)] is so small that in replacing sums over q by integrals, the infinite volume limit is not achieved even when the electron gas fills the universe. By repeating the calculation with the sums (4.25) and not the integrals (4.27), one can show that $(E-E_0)/V$ is proportional to g/L_0^2. For any physical system, the limit of infinite volume is never reached, and the energy reduction goes as the inverse square of the thickness of the sample along the magnetic field.

every pair of levels. This would result in an energy reduction containing a sum of terms like (4.30), but each term would even be negligible compared to the already negligible contribution from the two highest levels. One can also experiment with various more complicated forms of coupling than (4.3) but in all cases we have examined we always find an energy reduction proportional to $\exp[-4\pi(137)A]$, where A is of order unity or greater at any reasonable density.

The instability therefore appears to be a theoretical curiosity rather than an observable effect. The peculiar nonuniform ground state that it reflects is quite similar to the Hartree ground state with spatially varying density found by Overhauser for one-dimensional fermions with arbitrarily weak attractive interactions. The presence of a static magnetic field is necessary to give a one-dimensional nature to the density of states.[12] Our state has a spatially varying current instead of Overhauser's density wave, because it is the current-current interaction that provides the attraction. (This also explains why our coupling of states is between cylinders with $\Delta n=\pm 1$, rather than a more straightforward coupling within each cylinder: The latter coupling cannot lead to a state with volume currents.) For weak coupling, the Overhauser kind of state reduces the energy per particle by an amount proportional to $\exp(-1/g)$; because the magnetic current-current interaction is so small at nonrelativistic velocities, the energy of our state differs negligibly from that of the ordinary state.

The instability in the helicon dispersion relation can thus be briefly characterized as follows: The quantization of orbits by the magnetic field gives a one-dimensional nature to the density of states, and this, along with the attractive electron current-current interaction, leads to an Overhauser type of instability; however, because the only attractive interaction is a very weak magnetic one, the Overhauser state has negligibly lower energy than the ordinary state; consequently, the instability is only present at unattainably low temperatures.

ACKNOWLEDGMENT

We have had several helpful discussions with Dr. L. J. Sham.

APPENDIX

There is a classical stability theorem due to Newcomb[2] which follows from so general an argument that its failure in the quantum case is rather surprising. In this Appendix, we extend Newcomb's argument to cases (a)–(c) of the Introduction, and show why the argument breaks down for a quantum electron gas in a uniform magnetic field interacting through the full set of Maxwell's equations.

We begin by reviewing Newcomb's proof in a form slightly generalized to cover case (a), in which the electronic distribution function has the form

$$f(\mathbf{r},\mathbf{v},t)=f_0(v)+f_1(\mathbf{r},\mathbf{v},t)\,,\quad f_0(v)=[\exp(\beta(\tfrac{1}{2}mv^2-\mu))+1]^{-1}. \tag{A1}$$

The time development of f is given by the Boltzmann-Vlasov equation

$$(\partial/\partial t+\mathbf{v}\cdot\nabla)f(\mathbf{r},\mathbf{v},t)=-(e/m)\times[\mathbf{E}_1+(\mathbf{v}/c)\times(\mathbf{B}_0+\mathbf{B}_1)]\cdot\nabla_{\mathbf{v}}f(\mathbf{r},\mathbf{v},t)\,, \tag{A2}$$

where $\mathbf{B}_0$ is the uniform static magnetic field, and $\mathbf{E}_1$ and $\mathbf{B}_1$ are self-consistent fields given by the solution to Maxwell's equations with sources

$$\mathbf{j}_1(\mathbf{r},t)=e(m/2\pi)^3\int d\mathbf{v}\mathbf{v}f_1(\mathbf{r},\mathbf{v},t)\,,\quad \rho_1(\mathbf{r},t)=e(m/2\pi)^3\int d\mathbf{v}f_1(\mathbf{r},\mathbf{v},t)\,. \tag{A3}$$

When the disturbance from equilibrium is small we replace (A2) by the linearized equation

$$(\partial/\partial t+\mathbf{v}\cdot\nabla)f_1(\mathbf{r},\mathbf{v},t)=-(e/m)[\mathbf{E}_1\cdot\nabla_{\mathbf{v}}f_0(v)-\mathbf{v}\times\mathbf{B}_0\cdot\nabla_{\mathbf{v}}f_1(\mathbf{r},\mathbf{v},t)]\,. \tag{A4}$$

Newcomb's theorem is that there is no solution to (A4), (A3), and Maxwell's equations, in which $f_1(\mathbf{r},\mathbf{v},t)$ has an unbounded growth in time for any $\mathbf{r}$ and $\mathbf{v}$. This is proved by observing that the function

$$\Delta F=\int d\mathbf{r}(E_1^2+B_1^2)/8\pi+\tfrac{1}{2}k_BT(m/2\pi)^3\int d\mathbf{r}d\mathbf{v}[f_1^2/f_0(1-f_0)] \tag{A5}$$

is a constant of the motion as a result of (A4), (A3), and Maxwell's equations.[13] But if $f_1(\mathbf{r},\mathbf{v},t)$ were to grow in time without bound, then so would the second term in ΔF, since $f_1^2/f_0(1-f_0)$ is everywhere positive. Since the first term in ΔF is also positive, ΔF would have to grow in time, which contradicts $\partial\Delta F/\partial t=0$.

The trick of producing this kind of proof lies in finding a function like ΔF having a negative or vanishing time derivative, and consisting of an appropriate sum of positive terms. Such a function, in nonlinear stability theory, is called a Lyapunov function.[14] Newcomb's Lyapunov

[12] This effect of the static magnetic field might enhance the Overhauser spin-wave instability [A. W. Overhauser, Phys. Rev. Letters 4, 466 (1960)] to the point where it could give observable effects in three dimensions even for a relatively weak exchange coupling. This question is now being investigated.

[13] We consider (without loss of generality since the equations are linear) f_1 to have a periodic space dependence with wave vector $\mathbf{k}$, and take all space integrals over an integral number of periods; this enables us to discard the terms $\int\mathbf{B}_0\cdot\mathbf{B}_1$ and $\int(\mathbf{E}_1\times\mathbf{B}_1)$, which can be transformed into vanishing surface integrals.

[14] Applications of Lyapunov's method to plasmas are discussed by T. K. Fowler, J. Math. Phys. 4, 559 (1963).

function can be derived from the free energy in the grand canonical ensemble

$$F=\int d\mathbf{r}[E_1^2+(\mathbf{B}_0+\mathbf{B}_1)^2]/8\pi+(m/2\pi)^3\int d\mathbf{r}d\mathbf{v} \times[(\tfrac{1}{2}mv^2-\mu)f-k_BT(f\ln f+(1-f)\ln(1-f)]. \quad \text{(A6)}$$

To second order in f_1, $F=F_0+\Delta F$. There is no *a priori* reason why such an expansion of the free energy should produce a Lyapunov function; indeed Newcomb's proof breaks down quantum mechanically just because the second-order free energy is no longer such a function.

To construct a quantum version of Newcomb's argument, we start from the equation of motion for the one-particle density matrix

$$i\partial\varphi/\partial t=[\mathcal{H},\varphi], \quad \text{(A7)}$$

where

$$\mathcal{H}(t)=\tfrac{1}{2}m(\mathbf{v}_0-e\mathbf{A}_1(t)/mc)^2.$$

$\mathbf{A}_1$ is the self-consistent vector potential (in the gauge in which the scalar potential vanishes) satisfying Maxwell's equations with sources

$$\begin{aligned}\mathbf{j}(\mathbf{r},t)&=\tfrac{1}{2}e\langle\mathbf{r}|\{\varphi(t),\mathbf{v}_0-(e/mc)\mathbf{A}_1(t)\}|\mathbf{r}\rangle,\\ \rho(\mathbf{r},t)&=e(\langle\mathbf{r}|\varphi(t)|\mathbf{r}\rangle-n_0).\end{aligned} \quad \text{(A8)}$$

(We use the same notation as in Sec. IV.) If $\varphi=\varphi^0+\varphi^1$, and we linearize about the equilibrium

$$\varphi^0=[e^{\beta(\mathcal{H}_0-\mu)}+1]^{-1},$$

then φ^1 obeys

$$i\partial\varphi^1/\partial t=[\mathcal{H}_0,\varphi^1]+[\mathcal{H}_1,\varphi^0], \quad \text{(A9)}$$

where

$$\mathcal{H}_1=-(e/2c)\{\mathbf{v}_0,\mathbf{A}_1\} \quad \text{(A10)}$$

and $\mathbf{A}_1$ satisfies Maxwell's equations with sources

$$\begin{aligned}\mathbf{j}(\mathbf{r},t)&=\tfrac{1}{2}e\langle\mathbf{r}|\{\varphi^1(t),\mathbf{v}_0\}|\mathbf{r}\rangle-(e^2/mc)\langle\mathbf{r}|\varphi_0\mathbf{A}_1(t)|\mathbf{r}\rangle,\\ \rho(\mathbf{r},t)&=e\langle\mathbf{r}|\varphi^1(t)|\mathbf{r}\rangle.\end{aligned} \quad \text{(A11)}$$

The free energy is

$$F=\int d\mathbf{r}[E_1^2+(\mathbf{B}_0+\mathbf{B}_1)^2]/8\pi+\mathrm{tr}(\varphi(\mathcal{H}-\mu) +k_BT\,\mathrm{tr}[\varphi\ln\varphi+(1-\varphi)\ln(1-\varphi)]. \quad \text{(A12)}$$

To second order this can be written $F=F_0+\Delta F$, with

$$\Delta F=\int d\mathbf{r}(E_1^2+B_1^2)/8\pi+\tfrac{1}{2}\sum_{\alpha\beta}|\langle\alpha|\varphi^1|\beta\rangle|^2 \times(\epsilon_\beta-\epsilon_\alpha)/(\varphi_\alpha{}^0-\varphi_\beta{}^0)+\mathrm{tr}(\varphi^0e^2A_1{}^2/2mc^2) -\mathrm{tr}(\varphi^1e\{\mathbf{v}_0,\mathbf{A}_1\}/2c), \quad \text{(A13)}$$

where the $|\alpha\rangle$ and $|\beta\rangle$ are complete sets of eigenstates of $\mathcal{H}_0$, ϵ_α is the eigenvalue of $\mathcal{H}_0$ in the state $|\alpha\rangle$, and $\varphi_\alpha{}^0=1/(e^{\beta(\epsilon_\alpha-\mu)}+1)$. It follows from (A13), (A11), (A9), and Maxwell's equations that $\partial\Delta F/\partial t=0$. What now cannot be established in general is that ΔF has the positiveness properties of a Lyapunov function. By completing the square for the terms involving φ^1, one can write

$$\Delta F=\int d\mathbf{r}(E_1^2/8\pi)+R(\mathbf{A}_1)+\tfrac{1}{2}\sum_{\alpha\beta}\left|\langle\alpha|\varphi^1|\beta\rangle -(e/2c)\frac{\varphi_\alpha{}^0-\varphi_\beta{}^0}{\epsilon_\beta-\epsilon_\alpha}\langle\alpha|\{\mathbf{v}_0,\mathbf{A}_1\}|\beta\rangle\right|^2\frac{\epsilon_\beta-\epsilon_\alpha}{\varphi_\alpha{}^0-\varphi_\beta{}^0}, \quad \text{(A14)}$$

where

$$R(\mathbf{A})=\int d\mathbf{r}\frac{(\nabla\times\mathbf{A})^2}{8\pi}+\mathrm{tr}\frac{\varphi_0e^2A^2}{2mc^2} -\frac{e^2}{2c^2}\sum_{\alpha\beta}\frac{\varphi_\alpha{}^0-\varphi_\beta{}^0}{\epsilon_\beta-\epsilon_\alpha}|\langle\alpha|\tfrac{1}{2}\{\mathbf{v}_0,\mathbf{A}\}|\beta\rangle|^2. \quad \text{(A15)}$$

We have separated from ΔF the quantity $R(\mathbf{A}_1)$, which is a quadratic functional of $\mathbf{A}_1$ independent of φ^1. If $R(\mathbf{A})$ were positive semidefinite, then the quantum version of Newcomb's theorem would be established, for, since $(\epsilon_\beta-\epsilon_\alpha)/(\varphi_\alpha{}^0-\varphi_\beta{}^0)>0$, if any matrix element of φ^1 grew without bound, no compensating growth of $\mathbf{A}_1$ could occur that would not result in ΔF also growing.

Thus, Newcomb's argument in the quantum case leads not to a proof of stability, but to a sufficient condition for stability. One can show that in the classical limit $R(\mathbf{A})=\int d\mathbf{r}B_1^2/8\pi$, which is positive definite. We can also show that $R(\mathbf{A})$ is positive semidefinite in cases (b) and (c). For this purpose, we first verify that $R(\mathbf{A})$ is gauge invariant. This would follow if we could show that

$$0=\mathrm{tr}(\varphi^0e^2\mathbf{A}\cdot\nabla\Lambda/2mc^2)-(e^2/2c^2) \times\sum_{\alpha\beta}\langle\alpha|\tfrac{1}{2}\{\mathbf{v}_0,\nabla\Lambda\}|\beta\rangle\langle\beta|\tfrac{1}{2}\{\mathbf{v}_0,\mathbf{A}\}|\alpha\rangle \times(\varphi_\alpha{}^0-\varphi_\beta{}^0)/(\epsilon_\beta-\epsilon_\alpha), \quad \text{(A16)}$$

for arbitrary functions $\mathbf{A}(\mathbf{r})$ and $\Lambda(\mathbf{r})$. Now $\tfrac{1}{2}\{v_0,\nabla\Lambda\}=(1/i)[\Lambda,\mathcal{H}_0]$, so the second term in (A16) is just

$$\begin{aligned}&i(e^2/2c^2)\sum_{\alpha\beta}(\varphi_\alpha{}^0-\varphi_\beta{}^0)\langle\alpha|\Lambda|\beta\rangle\langle\beta|\tfrac{1}{2}\{\mathbf{v},\mathbf{A}\}|\alpha\rangle\\ &=i(e^2/4c^2)\,\mathrm{tr}\,\varphi^0[\Lambda,\{\mathbf{v}_0,A\}]=i(e^2/4c^2)\,\mathrm{tr}\,\varphi^0\{[\mathbf{v}_0,\Lambda],\mathbf{A}\}\\ &=(e^2/2mc^2)\,\mathrm{tr}\,\varphi^0\mathbf{A}\cdot\nabla\Lambda,\end{aligned} \quad \text{(A17)}$$

which cancels the first.

In particular, $R(\mathbf{A})$ vanishes for a purely longitudinal $\mathbf{A}$, and this establishes stability in the limit where only Coulomb interactions are retained.[15] It also follows from gauge invariance that the general sufficient condition for stability can be reduced to $R(\mathbf{A})$ being non-negative for all transverse vector fields $\mathbf{A}$. This condition is met when $\mathbf{B}_0=0$, for then the eigenstates of $\mathcal{H}_0$ are just plane waves, and the last two terms of $R(\mathbf{A})$ are, for any trans-

[15] This result has been proved directly from the dispersion relation by N. D. Mermin and E. Canel, Ann. Phys. (N. Y.) **26**, 247 (1964).

verse $\mathbf{A}$, proportional to the following expression, where $p_\perp$ is the component of $\mathbf{p}$ perpendicular to $\mathbf{k}$:

$$(e^2/2mc^2)\sum_{\mathbf{k}}|\mathbf{A}(\mathbf{k})|^2\int(d\mathbf{p}/(2\pi)^3)[F(p^2/2m)-(p_\perp^2/2\mathbf{p}\cdot\mathbf{k})[F((\mathbf{p}-\tfrac{1}{2}\mathbf{k})^2/2m)-F((\mathbf{p}+\tfrac{1}{2}\mathbf{k})^2/2m)]]. \quad \text{(A18)}$$

Now

$$\int d\mathbf{p}(p_\perp^2/2\mathbf{p}\cdot\mathbf{k})[F((\mathbf{p}-\tfrac{1}{2}\mathbf{k})^2/2m)-F((\mathbf{p}+\tfrac{1}{2}\mathbf{k})^2/2m)]=-\tfrac{1}{2}\int d\mathbf{p}(p_\perp^2/2m)\int_{1}^{1}d\alpha F'((p^2+\alpha\mathbf{p}\cdot\mathbf{k}+k^2/4)/2m)$$

$$=-\tfrac{1}{2}\int_{-1}^{1}d\alpha\int d\mathbf{p}(p_\perp^2/2m)\partial F((p^2+(1-\alpha^2)k^2/4)/2m)/\partial(p_\perp^2/2m)$$

$$=\tfrac{1}{2}\int d\mathbf{p}\int_{-1}^{1}d\alpha F((p^2+k^2(1-\alpha^2)/4)/2m)<\int d\mathbf{p}F(p^2/2m), \quad \text{(A19)}$$

so (A18) is positive.

We have therefore established stability in cases (a), (b), and (c). However, in the presence of a magnetic field, $R(\mathbf{A})$ is not positive semidefinite. In particular, if $\mathbf{A}(\mathbf{r})$ is transverse and varies only along the direction of $\mathbf{B}_0$,

$$\mathbf{A}(\mathbf{r})=\sum_k \mathbf{A}(k)e^{ikz},$$

then

$$R(\mathbf{A})=(V/8\pi)\sum_k(k^2+\omega_p^2/c^2)|\mathbf{A}(k)|^2$$

$$-(e^2/2c^2)\sum_{\alpha\beta}\left|\sum_k\int d\mathbf{r}\psi_\alpha^*(\mathbf{r})(A^+(k)v^-+A^-(k)v^+)e^{ikz}\psi_\beta(\mathbf{r})\right|^2(\varphi_\alpha^0-\varphi_\beta^0)/(\epsilon_\beta-\epsilon_\alpha). \quad \text{(A20)}$$

With (4.14) and (4.1), the matrix elements in (A20) are easily evaluated; we find (with a suitable choice for the phases of $A^{(\pm)}$),

$$R(\mathbf{A})=(V/4\pi c^2)\sum_k|\mathbf{A}(k)|^2\Big[k^2c^2+\omega_p^2-(m\omega_c^2\omega_p^2/4\pi^2n_0)\sum_n(n+1)$$

$$\times\int dp[f_n(p-\tfrac{1}{2}k)-f_{n+1}(p+\tfrac{1}{2}k)]/(\omega_c+pk/m)\Big]. \quad \text{(A21)}$$

But the quantity in brackets [see (2.2)] has been shown in Sec. III to be negative at low temperatures for certain values of k. Thus, Newcomb's argument does not produce a stability theorem in the general case, and is not inconsistent with the instability in the quantum helicon dispersion relation.